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A DYNAMIC SOLUTION CONCEPT FOR ABSTRACT GAMES

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ABSTRACT

Several solution concepts have been defined for abstract games. Some of these are the core due to Gillies [3] and Shapley, the von Neumann-Morgenstern stable sets due to von Neumann and Morgenstern [15], and the subsolutions due to Roth [11]. These solutions concepts are rather static in nature. In this paper, we propose a new solution concept for abstract games called the dynamic solution that reflects the dynamic aspects of negotiation among the players. Some properties of the dynamic solution are studied. Also, the dynamic solution of abstract games arising from n-person cooperative games in characteristic function form is investigated.

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SIGNIFICANCE AND EXPLANATION

Abstract games are abstractions of games that arise in game theory, social choice theory, economic market theory, coalition theory, theory of legislatures and committees and various other situations that can be modeled as a game.

A fundamental problem arising here is to predict the outcome(s) which will result if the game is played by rational players. These predictions are called solutions of the abstract game. Several solution concepts have been defined for abstract games. However, these are somewhat static in nature and do not indicate the dynamics of negotiations or the mechanism by which the outcomes in the solution are realized. In this paper, we propose and study a new solution concept called the dynamic solution, that reflects the dynamic aspects of bargaining among the players. The dynamic solution is based on the elementary theory of Markov chains.

Static solution concepts tend to be normative, i.e., given certain assumptions, they tell people how to behave to attain certain ends. The dynamic solution concept tends to be descriptive, i.e., given certain assumptions, it predicts how people will behave. This is illustrated by several examples.

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A DYNAMIC SOLUTION CONCEPT FOR ABSTRACT GAMES

Prakash P. Shenoy

1. Introduction.

Several solution concepts have been defined for abstract games. Some of these are the core due to Gillies [3] and Shapley, the von Neumann-Morgenstern stable sets due to von Neumann and Morgenstern [15], and the subsolutions due to Roth [11]. These solution concepts are rather static in nature. They do not indicate the dynamics of negotiation or the mechanism by which the outcomes in the solution are realized in practice. They can be considered as (conditionally) normative or prescriptive theories. In this paper, we propose a new solution concept for abstract games, called the dynamic solution, that reflects the dynamic aspects of bargaining among the players. It is based on the elementary theory of Markov chains.

Section 2 contains some notation and definitions. We introduce two additional binary relations which are derived from the binary relation, domination, of the abstract game. In Section 3, the concepts of an elementary dynamic solution and the dynamic solution are introduced and discussed. The properties of the dynamic solution are studied in Section 4. For an abstract game with a finite number of outcomes, the concept of the dynamic solution coincides with the concept of the R-admissible set defined by Kalai, Pazner and Schmeidler [4].† In Section 5, the dynamic solution of all abstract games arising from 3-person cooperative

† Kalai and Schmeidler [5] have also defined a solution concept similar to the dynamic solution for infinite outcomes abstract games. However, the research presented here was done independently of both these references.

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games with side payments in characteristic function form are determined. Finally, in Section 6, many games, which have pathological behaviour in the classical von Neumann-Morgenstern (vN-M) theory of stable sets, are shown to be amenable to our approach.

2. Notation and Definitions

An abstract game is a pair (X, dom) where X is an arbitrary set whose members are called outcomes of the game, and dom is an arbitrary binary relation defined on X and is called domination. An outcome $x \in X$ is said to be accessible from an outcome $y \in X$, denoted by $x \leftarrow y$ (or $y \rightarrow x$), if there exists outcomes $z_0 = x, z_1, z_2, \ldots, z_{m-1}, z_m = y$, where m is a positive integer, such that

$$(1) \qquad x = z_0 \text{ dom } z_1 \text{ dom } z_2 \text{ dom } \ldots \text{ dom } z_{m-1} \text{ dom } z_m = y.$$

Also assume $x \leftarrow x$, i.e. an *outcome* is accessible from itself. Clearly, the binary relation accessible is transitive and reflexive.

An interpretation of the relation accessible is as follows: If the players are considering an outcome y at some stage, then an outcome they will consider next will be a $z \in X$ such that z dom y. If $x \leftarrow y$ and if the players are considering outcome y at some time, then it is possible that they will consider outcome x at some future time. I.e. one may interpret the relation as a possible succession of transitions from one outcome to another.

Two outcomes x and y which are accessible to each other are said to communicate and we write this as $x \leftrightarrow y$. Since the relation accessible is transitive and reflexive it follows that

Proposition 1. Communication is an equivalence relation.

We can now partition the set X into <u>equivalence classes</u>. Two outcomes are in the same equivalence class if they communicate with each other. We say that the abstract game is <u>irreducible</u> if this equivalence relation induces only one class. The set

(2) $$\mathrm{Dom}(x) = \{y \in X : x \text{ dom } y\}$$

is called the <u>dominion</u> of x. Similarly we define the dominion of any subset $A \subset X$ by

(3) $$\mathrm{Dom}(A) = \bigcup_{x \in A} \mathrm{Dom}(x)$$

and let

(4) $$\mathrm{Dom}^k(A) = \mathrm{Dom}(\mathrm{Dom}^{k-1}(A)) \quad \text{for } k \geq 2$$

$$\text{where } \mathrm{Dom}^1(A) = \mathrm{Dom}(A) .$$

Also define the <u>inverse dominion</u> of x by

(5) $$\mathrm{Dom}^{-1}(x) = \{y \in X : y \text{ dom } x\}.$$

The <u>core</u> C (due to Gillies [3] and Shapley) of an abstract game is defined to be the set of undominated outcomes. I.e.

(6) $$C = X - \mathrm{Dom}(X) .$$

We can rewrite the definition of the core in terms of the relation accessible as follows:

(7) $$C = \{x \in X : \text{ For all } y \in X,\ y \neq x, \text{ we have } y \not- x\},$$

i.e., in the terminology of Markov chains, the core is the set of all <u>absorbing</u> outcomes. Note that each outcome in the core (if non-empty) is an equivalence class by itself.

A <u>vN-M stable set</u> V (due to von Neumann and Morgenstern [15]) of an abstract game is any $V \subset X$ such that

(8) $$V = X - \mathrm{Dom}(V) .$$

Any vN-M stable set V satisfies internal stability and external stability, i.e.,

(9) $V \cap \mathrm{Dom}(V) = \emptyset$ and $V \cup \mathrm{Dom}(V) = X$.

In recent years, Behzad and Harary [1,2] and Shmadich [13] have characterized finite abstract games for which vN-M stable sets exist.

3. The Dynamic Solution.

We define an elementary dynamic solution (elem. d-solution) of the abstract game (X, dom) as a set $S \subset X$ such that

(10) if $x \in S$, $y \in X-S$, then $y \not\leftarrow x$ and

(11) if $x, y \in S$, then $x \leftarrow y$ and $y \leftarrow x$.

Condition (10) requires S to be 'externally stable' in a dynamic sense, i.e. if the players are considering $x \in S$ at some time, then they will never consider any outcome that is not in S in the future. We can think of Condition (11) as 'internal stability' in a dynamic sense. I.e., if the players make a transition (in the consideration of outcomes) from x to y then it is possible that the players will again consider the outcome x in the future.

Proposition 2. An elem. d-solution S is an equivalence class.

Proof: By Condition (11), S is contained in an equivalence class H, i.e. $S \subset H$. Suppose $S \neq H$. Let $x \in H-S$ and $y \in S$. Then $x \leftarrow y$ since H is an equivalence class, which contradicts (10). □

The converse, however, is not always true, i.e., an equivalence class need not be an elem. d-solution. Condition (10) requires S to be (in the terminology of Markov chains) a non-transient (recurrent, persistent) equivalence class.

Proposition 3. Each outcome in the core C of the game is an elem. d-solution.

The proof follows from the definition of the core in (7).

The dynamic solution (d-solution) P of the game is the union of all distinct elementary dynamic solutions. I.e.

(12) $$P = \cup\{S \subset X: S \text{ is an elem. d-solution.}\}$$

We can interpret P as the set of all likely outcomes of the game.

Proposition 4. For any abstract game, the dynamic solution always exists and is unique. However, it may be empty.

Proof: Existence follows from the fact that the empty set $\emptyset$ is always an elem. d-solution. Uniqueness is clear from Proposition 2 and the definition of the d-solution. □

Proposition 5. $C \subset P$

The proof follows from Proposition 3 and the definition of P.

4. Properties of the Dynamic Solution

If X is a finite set, then our definition of the d-solution coincides with the definition of the R-admissible set due to Kalai, Pazner and Schmeidler [4]. In this section we demonstrate the equivalence of the two definitions. This will also illustrate some of the properties of the d-solution.

Lemma 6. If X is a finite set, then P is the d-solution if and only if P satisfies:

(13) For all $x,y \in P$, $y \leftarrow x \Leftrightarrow x \leftarrow y$.

(14) If $x \in P$, $y \in X-P$, then $y \not\leftarrow x$. And

(15) if $y \in X-P$, then $\exists\, x \in P$ such that $x \leftarrow y$.

Proof: (Necessity): It is clear from the definition of P that it satisfies Conditions (13) and (14). Suppose Condition (15) does not hold. Then for some $y_1 \in X-P$, $x \not\leftarrow y_1$ for all $x \in P$. Let $A_1(y_1) \subset X-P$ be the equivalence class containing y_1. If $A_1(y_1)$ satisfies Condition (10), then $A_1(y_1)$ is an elem. d-solution which is a contradiction. If not, then $\exists\, y_2 \in X-P-A_1(y_1)$ such that $y_2 \leftarrow x$ for some $x \in A_1(y_1)$. Let $A_2(y_2) \subset X-(P \cup A_1(y_1))$ be the equivalence class containing y_2. Repeating this argument, since X is finite, we get an equivalence class $A_k(y_k) \subset X - P - \bigcup_{i=1}^{k-1} A_i(y_i)$ satisfying Condition (10). Hence $A_k(y_k)$ is an elem. d-solution, which is a contradiction!

(Sufficiency): Statements (13) and (14) imply that P is a union of elem. d-solutions. Suppose some elem. d-solution S is not included in P, and let $y \in S \subset X-P$. Then from Condition (15) $\exists\, x \in P$ such that $x \leftarrow y$. But $x \notin S$ contradicts the fact that S is an elem. d-solution! Hence P is the union of all elem. d-solutions. □

Theorem 7. If X is a finite set, then the d-solution is nonempty and unique.

Proof: Nonemptiness follows from Condition (15) of Lemma 6. Uniqueness follows from Proposition 4. □

Remark: If R is an arbitrary binary relation defined on X, Kalai, Pazner and Schmeidler define an R-admissible set as a subset of X satisfying Conditions (13), (14) and (15) with the binary relation R substituted in place of $\leftarrow$.

Define a binary relation transitive-domination denoted by t-dom as follows:

(16) For all $x,y \in X$, x t-dom $y \Leftrightarrow x \leftarrow y$ and $y \not\leftarrow x$.

Transitive domination is asymmetric and transitive. The following lemma is proved in Kalai, Pazner and Schmeidler [4].

Lemma 8. If X is a finite set, the d-solution P satisfies:

(17) For all $x,y \in P$, x t$\not$dom y and y t$\not$dom x (internal stability).

(18) For all $y \in X-P$, $\exists\, x \in P$ such that x t-dom y (external stability).

I.e. P is the unique vN-M stable set and the core of the abstract game (X,t-dom).

The following results are easy consequences of the definition of the d-solution. Nevertheless, they are useful in computing the d-solution.

Proposition 9. If $x,y \in X$ such that $x \leftarrow y$ and $y \not\leftarrow x$, then $y \notin P$.

Proof: If $x \in P$, then $y \in P$ contradicts Condition (13). If $x \notin P$, then $y \in P$ contradicts Condition (14). Note that Conditions (13) and (14) hold for infinite abstract games also. □.

Corollary 10. Let y be an outcome that is not in the core. Then $\mathrm{Dom}(y) = \emptyset \Rightarrow y \notin P$.

Proposition 11. $x \notin P \Rightarrow \mathrm{Dom}^k(x) \cap P = \emptyset$ for all integers $k \geq 1$.

Proof: $y \in \mathrm{Dom}^k(x) \Rightarrow y \rightarrow x$ for $x \notin P \Rightarrow y \notin P$. □

Proposition 12. If the core C is the unique vN-M solution, then $P = C$.

Proof: From Proposition 5, $C \subset P$. Since C is the unique vN-M stable set, $y \in X-C \Rightarrow \exists\, x \in C$ such that $x \leftarrow y$. But $y \not\leftarrow x$ (since $x \in C$). Hence $y \notin P$ (by Proposition 9). □

Corollary 13. Let C be nonempty. If $y \in \mathrm{Dom}^k(C)$ for some integer $k \geq 1$ then $y \notin P$. I.e. $P \subset X - \bigcup_{j=1}^{m} \mathrm{Dom}^j(C)$ for every integer $m \geq 1$.

5. Dynamic Solutions of 3-Person Games

A cooperative n-person game with side payments in characteristic function form is a pair (N,v) where $N = \{1,2,\ldots,n\}$ denotes the set of players and v is a non-negative real valued function defined on the subsets of N which satisfies $v(\emptyset) = 0$ and $v(\{i\}) = 0$ for all $i \in N$. The subsets of N are called coalitions. A coalition structure (c.s.) $\mathcal{P} = \{P_1,\ldots,P_m\}$ is a partition of N into disjoint (nonempty) coalitions. The set of (payoff) outcomes corresponding to coalition structure $\mathcal{P}$ is denoted by $X(\mathcal{P})$, where

$$X(\mathcal{P}) = \{x \in E^n : x_i \geq 0 \text{ for all } i \in N \text{ and } \sum_{i \in P_j} x_i = v(P_j) \text{ for each } P_j \in \mathcal{P}\} \tag{19}$$

The elements of the set $X(\{N\})$ are referred to as imputations. Domination is defined as follows:

> $x \in X(\mathcal{P})$ is said to dominate $y \in X(\mathcal{P})$ via coalition R, denoted by $x \operatorname{dom}_R y$ if $x_i > y_i$ for all $i \in R$ and $\sum_{i \in R} x_i \leq v(R)$.
>
> x dominates y, denoted by $x \operatorname{dom} y$ if $\exists$ a nonempty $R \subset N$ such that $x \operatorname{dom}_R y$.

In the abstract game $(X(\mathcal{P}),\mathrm{dom})$ as defined above, we cannot have domination via N and via one player coalitions. Also, if $x_i = 0$, then x does not dominate any other outcome via coalitions that contain player i. Hence we have the following result.

Lemma 14. Let Γ be a 3-person game and P be a c.s. that contains only one-player or two-player coalitions. Then the dynamic solution of the game $(X(P),\mathrm{dom})$ is the entire set of outcomes, i.e. $P(P) = X(P)$.

So we need concern ourselves with only the c.s. $P = \{N\}$. Let $P(\{N\})$ and $C(\{N\})$ denote the dynamic solution and the core of the abstract game $(X(\{N\}),\mathrm{dom})$. To condense notation we will denote $P(\{N\})$ and $C(\{N\})$ by $P(N)$ and $C(N)$ respectively. Assume without loss of generality that the characteristic function satisfies

$$v(\{1,2\}) \leq v(\{1,3\}) \leq v(\{2,3\}).$$

Let $v(\{1,2\}) = a$, $v(\{1,3\}) = b$, $v(\{2,3\}) = c$ and $v(\{1,2,3\}) = d$. The following inclusive cases should be distinguished:

Case 1) $d \geq (a+b+c)/2$, $d \geq c$.

In this case the core $C(N) \neq \emptyset$ and is given by

$$C(N) = \{x \in E^3 : x_i \geq 0 \text{ for all } i \in N,\ x_1 + x_2 \geq a,$$
$$x_1 + x_3 \geq b,\ x_2 + x_3 \geq c \text{ and } x_1 + x_2 + x_3 = d\}.$$

The d-solution is given by $P(N) = C(N)$. (See Figure 1.)

Case 2) $d < (a+b+c)/2$, $d \geq c$.

In this case $C(N) = \emptyset$. The d-solution is given by

$$P(N) = \mathrm{Conv}\{w_1,w_2,w_3\} - \{w_1,w_2,w_3\} \text{ where}$$
$$w_1 = (a+b-d,\ d-b,\ d-a),$$
$$w_2 = (d-c,\ a+c-d,\ d-a),$$
$$w_3 = (d-c,\ d-b,\ b+c-d)$$

and $\mathrm{Conv}\{a_1,\ldots,a_p\}$ denotes the convex hull of the points in $\{a_1,\ldots,a_p\}$. (See Figure 2.)

Case 3) $a \le b \le d < c, \quad d \ge a+b$.

In this case the core $C(N) \ne \emptyset$ and is given by

$$C(N) = \mathrm{Conv}\{(0,a,d-a),\ (0,d-b,b)\}$$

and the d-solution is given by $P(N) = C(N)$. (See Figure 3.)

Case 4) $a \le b \le d < c, \quad d < a+b$.

In this case, $C(N) = \emptyset$. The d-solution is given by

$$P(N) = \mathrm{Conv}\{(a+b-d,\ d-b,\ d-a),\ (0,\ a,\ d-a),\ (0,\ d-b,\ b)\}$$
$$- \{(a+b-d,\ d-b,\ d-a),\ (0,\ a,\ d-a),\ (0,\ d-b,\ b)\}$$

(See Figure 4.)

Case 5) $a \le d < b \le c$.

In this case $C(N) = \emptyset$. The d-solution is given by

$$P(N) = \mathrm{Conv}\{(a,\ 0,\ d-a),\ (0,\ a,\ d-a),\ (0,\ 0,\ d)\}$$
$$- \{(a,\ 0,\ d-a),\ (0,\ a,\ d-a),\ (0,\ 0,\ d)\}$$

(See Figure 5.)

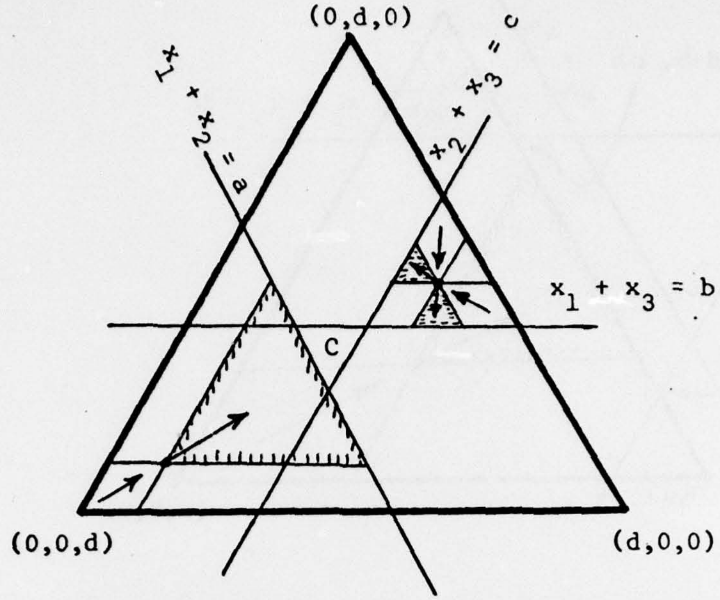
Case 6) $d < a \le b \le c$.

In this case $C(N) = \emptyset$. The d-solution is given by

$$P(N) = \mathrm{Conv}\{(d,\ 0,\ 0),\ (0,\ d,\ 0),\ (0,\ 0,\ d)\}$$
$$- \{(d,\ 0,\ 0),\ (0,\ d,\ 0),\ (0,\ 0,\ d)\}$$

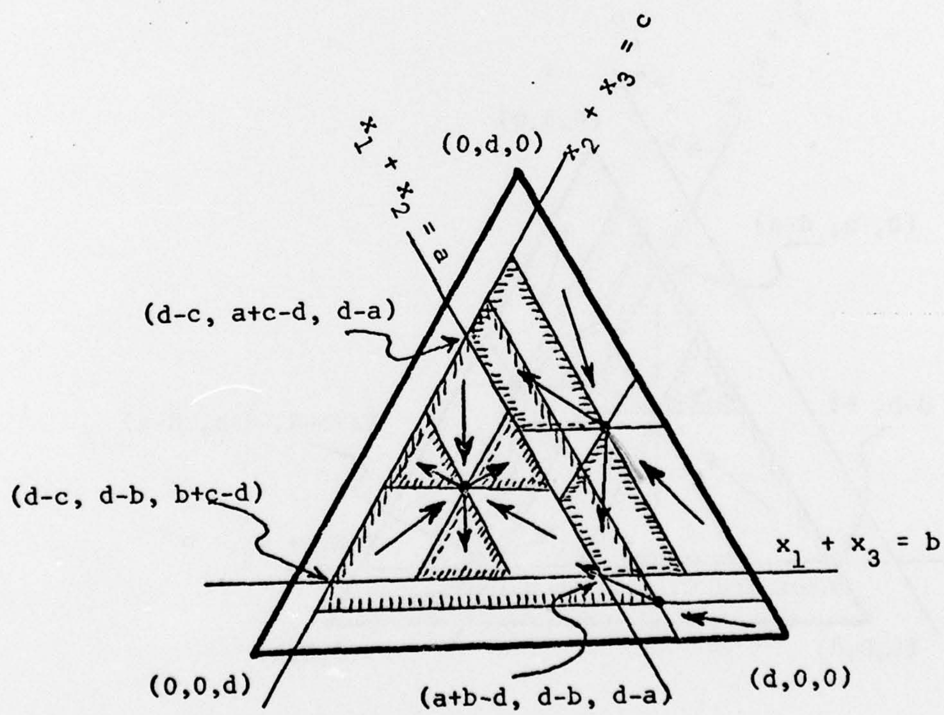
(See Figure 6.)

Thus all cases have been considered.



<u>Figure 1.</u> The dynamic solution P(N) of a 3-person game, Case 1)

The shaded region represents the inverse dominion of the point and the unshaded region(s) (containing the arrow) represent the dominion of the point. The arrows indicate the direction of transitions.



<u>Figure 2.</u> The dynamic solution P(N) of a 3-person game, Case 2).

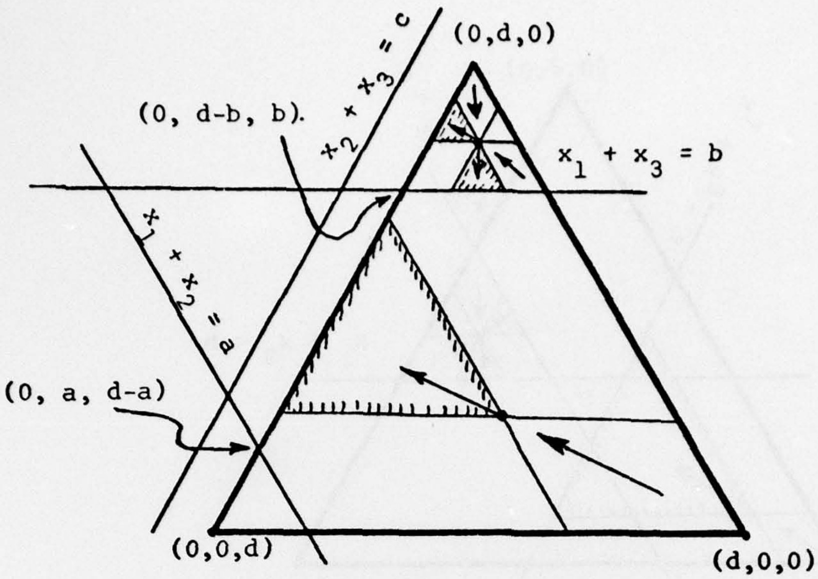


Figure 3. The dynamic solution P(N) of a 3-person game, Case 3).

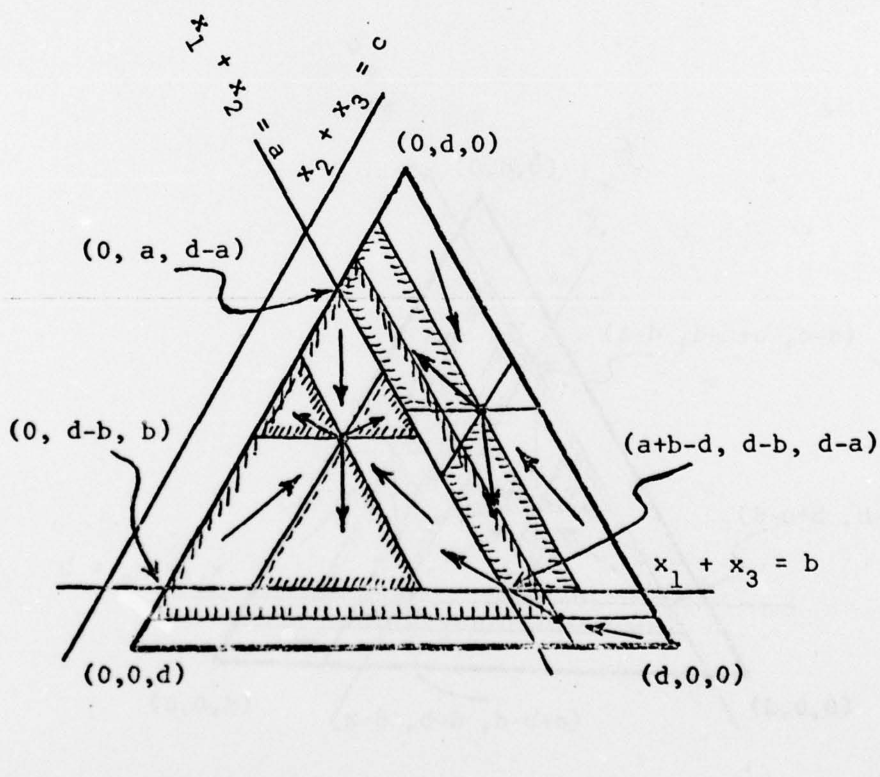
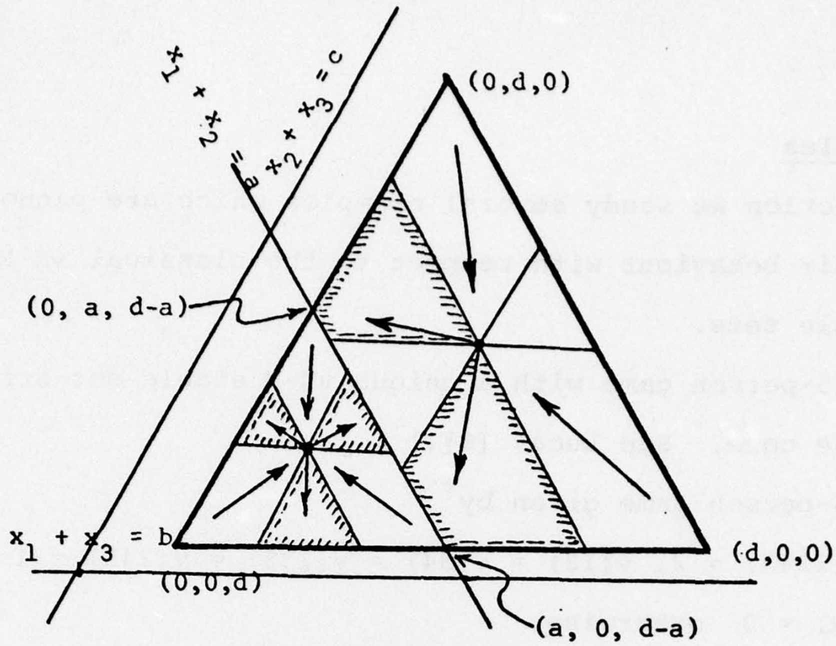
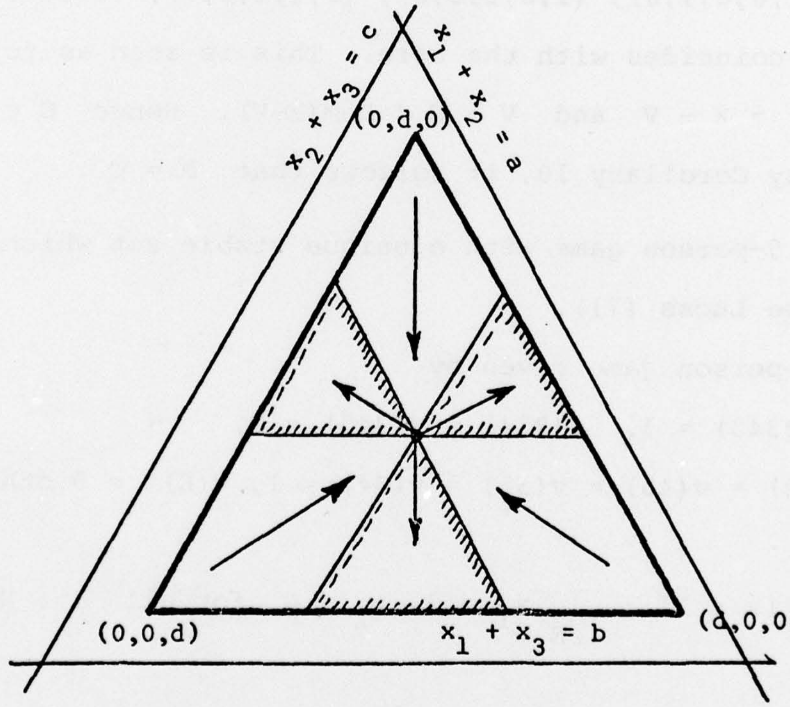


Figure 4. The dynamic solution P(N) of a 3-person game, Case 4).



<u>Figure 5.</u> The dynamic solution P(N) of a 3-person game, Case 5).



<u>Figure 6.</u> The dynamic solution P(N) of a 3-person game, Case 6).

6. Some Examples

In this section we study several examples which are pathological in their behaviour with respect to the classical vN-M theory of stable sets.

Example 1. (A 5-person game with a unique vN-M stable set strictly larger than the core. See Lucas [6].)

Consider the 5-person game given by†

$$v(12345) = 2,\ v(12) = v(34) = v(135) = v(245) = 1\ ,$$
$$v(R) = 0 \quad \text{otherwise.}$$

The core $C = \mathrm{Conv}\{(1,0,0,1,0),\ (0,1,1,0,0)\}$ and the unique vN-M solution is given by

$$V = \mathrm{Conv}\{(1,0,0,1,0),\ (1,0,1,0,0),\ (0,1,1,0,0),\ (0,1,0,1,0)\}.$$

The d-solution coincides with the core. This is seen as follows. We have $Dom(C) = X - V$ and $V - C \subset \mathrm{Dom}(X-V)$. Hence $C = X - \bigcup_{j=1}^{2} \mathrm{Dom}^j(C)$. By Corollary 10, it follows that $P = C$.

Example 2. (A 5-person game with a unique stable set which is non-convex. See Lucas [7]).

Consider the 5-person game given by

$$v(12345) = 3,\quad v(234) = v(345) = 2,$$
$$v(12) = v(45) = v(35) = v(34) = 1,\ v(R) = 0 \text{ otherwise.}$$

For this game

$$X = \{x \in E^5 : \sum_{i \in N} x_i = 3,\ x_i \geq 0 \quad \text{for all} \quad i \in N\}.$$

Let

$$B = \{x \in X : \sum_{i \in R} x_i \geq v(R) \quad \text{for all} \quad R \subset N \quad \text{except} \quad \{2,3,4\}\}.$$

†To condense notation we shorten expressions like $v(\{1,2,3,4,5\})$ to $v(12345)$.

Then the core C of the game is given by

$$C = \{x \in B : x_2 + x_3 + x_4 \geq 2\}.$$

It can be easily shown that $\mathrm{Dom}(C) \supset X - B$ and $B - C \subset \mathrm{Dom}(X-B)$. Hence by Corollary 10, we have $P = C$.

<u>Example 3.</u> (A game with no symmetric stable set. See Lucas [7].) Let $N = \{1,\ldots,8\}$, $v(N) = 4$, $v(1357) = 3$, $v(257) = v(457) = 1$, $v(12) = v(34) = v(56) = v(78) = 1$, $v(R) = 0$ for all other $R \subset N$. For this game,

$$X = \{x \in E^8 : \sum_{i \in N} x_i = 4, \text{ and } x_i \geq 0 \text{ for all } i \in N\}.$$

Let

$$H = \{x \in X : x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = 1\}.$$

Then the core C of the game is given by

$$C = \{x \in H : x_1 + x_3 + x_5 + x_7 \geq 3\}.$$

Define $F_i = \{x \in H : x_i = 1\}$ for $i = 1,3,5,7$, and

$$F = F_1 \cup F_3 \cup F_5 \cup F_7 - C.$$

It is shown in Lucas [7] that $\mathrm{Dom}(C) = (X-H) \cup (H - (C \cup F))$. It is also clear that $(H-C) \subset \mathrm{Dom}(X-H)$. Hence $C = X - (\mathrm{Dom}^1(C) \cup \mathrm{Dom}^2(C))$. By Corollary 13 it follows that $P = C$.

<u>Example 4.</u> (A game with no vN-M stable set. See Lucas [8,9].) Lucas [8] constructs a ten-person game in which the set of imputations can be partitioned into regions as follows:

$$X = (X-B) \cup (B - (C \cup E \cup F)) \cup (C \cup E \cup F)$$

where C is the nonempty core. The domination relations is such that

(20) $\mathrm{Dom}(C) \supset (X-B) \cup (B - (C \cup E \cup F))$,

(21) $F \cap \mathrm{Dom}(C \cup E \cup F) = \emptyset$,

(22) $E \subset \mathrm{Dom}(X-B)$.

By Corollary 13 and Relation (20), $P \subset (C \cup E \cup F)$, Relation (21) $\Rightarrow F \subset \mathrm{Dom}(\{X-B\} \cup \{B - (C \cup E \cup F)\}) \Rightarrow F \cap P = \emptyset$ using Corollary 13 and Relation (22) $\Rightarrow E \cap P = \emptyset$ by Corollary 13. Hence $P = C$.

Example 5. (An 8-person game with a unique stable set that is non convex. See Lucas [10]).

Let $N = \{1,...,8\}$, $v(N) = 4$, $v(1467) = 2$, $v(12) = v(34) = v(56) = v(78 = 1$, $v(R) = 0$ for all other $R \subset N$. For this game it can be shown as in Example 3 that $P = C$.

A game without side payments is a triple (N,v,X) where $N = \{1,...,n\}$ is a set of n players, v is a "generalized characteristic function" and X is the set of imputations. A generalized characteristic function v maps nonempty subsets of N into subsets of n-dimensional space E^n, where the subset $v(R)$ assigned to coalition R consists of all vectors x such that R can guarantee all of its members at least their share in X. We assume that v satisfies the following axioms for any nonempty $R \subset N$.

(23) $v(R)$ is closed, nonempty and convex.

(24) If $x \in v(R)$ and $y_i \leq x_i$ for all $i \in R$ then $y \in v(R)$.

(25) $v(R_1) \cap v(R_2) \subset v(R_1 \cup R_2)$ whenever $R_1 \cap R_2 = \emptyset$.

(26) $x \in v(N) \Leftrightarrow x_i \leq y_i$ for some $y \in X$ and for all $i \in N$.

Example 6. (A 7-person non side payment game with no vN-M stable sets. See Stearns [14].)

Let $N = \{1,...,7\}$ and X be the convex hull of the five imputations

$$p^1 = (1,1,2,0,0,0,0) \qquad c = (2,0,2,0,2,0,1)$$
$$p^2 = (0,0,1,1,2,0,0) \qquad o = (0,0,0,0,0,0,0)$$
$$p^3 = (2,0,0,0,1,1,0).$$

Let the "minimal winning" coalitions be

$$(135),\ (127),\ (347),\ (567)\ .$$

Note that a coalition is winning if it contains a minimal winning coalition as a subset. Define $v\colon 2^N - \emptyset \to E^7$ by

$$v(R) = \begin{cases} \{x \in E^7 : x_i \leq y_i \text{ for all } i \in R \text{ and for } \underline{\text{some}}\ y \in X & \text{when } R \text{ is winning} \\ \{x \in E^7 : x_i \leq y_i \text{ for all } i \in R \text{ and for } \underline{\text{all}}\ y \in X & \text{when } R \text{ is non-winning.} \end{cases}$$

The core of this game is the single imputation c. The d-solution is $P = C$. This is seen as follows.

$$\mathrm{Dom}(c) = X - (L_1 \cup L_2 \cup L_3) \quad \text{where } L_i = [c,p^i]$$

the closed line segment joining c and p^i for $i = 1,2,3$. Let $x \in L_i - c$, i.e., $x = (\lambda+1,\ 1-\lambda,\ 2,\ 0,\ 2\ ,\ 0,\ \lambda)$ for some $0 \leq \lambda < 1$. Let

$$y^1 = (2,\ 0,\ 2\lambda^1,\ 0,\ \lambda^1+1,\ 1-\lambda^1,\ \lambda^1) \quad \text{where } \lambda < \lambda^1 < 1,$$
$$y^2 = (2\lambda^2,\ 0,\ \lambda^2+1,\ \lambda^2-1,\ 2,\ 0,\ \lambda^2) \quad \text{where } \lambda^1 < \lambda^2 < 1 \text{ and}$$
$$y^3 = (\lambda^3+1,\ 1-\lambda^3, 2,\ 0,\ 2\lambda^3,\ 0,\ \lambda^3) \quad \text{where } \lambda^2 < \lambda^3 < 1\ ,$$

Then $y^3 \mathrm{dom}_{(127)}\ y^2\ \mathrm{dom}_{(347)}\ y^1\ \mathrm{dom}_{(567)}\ x$. Therefore $y^3 \prec x$. But $x \neq y^3$. Hence by Proposition 9, $x \notin P$. Hence $P = C = \{c\}$. (See Figure 7.)

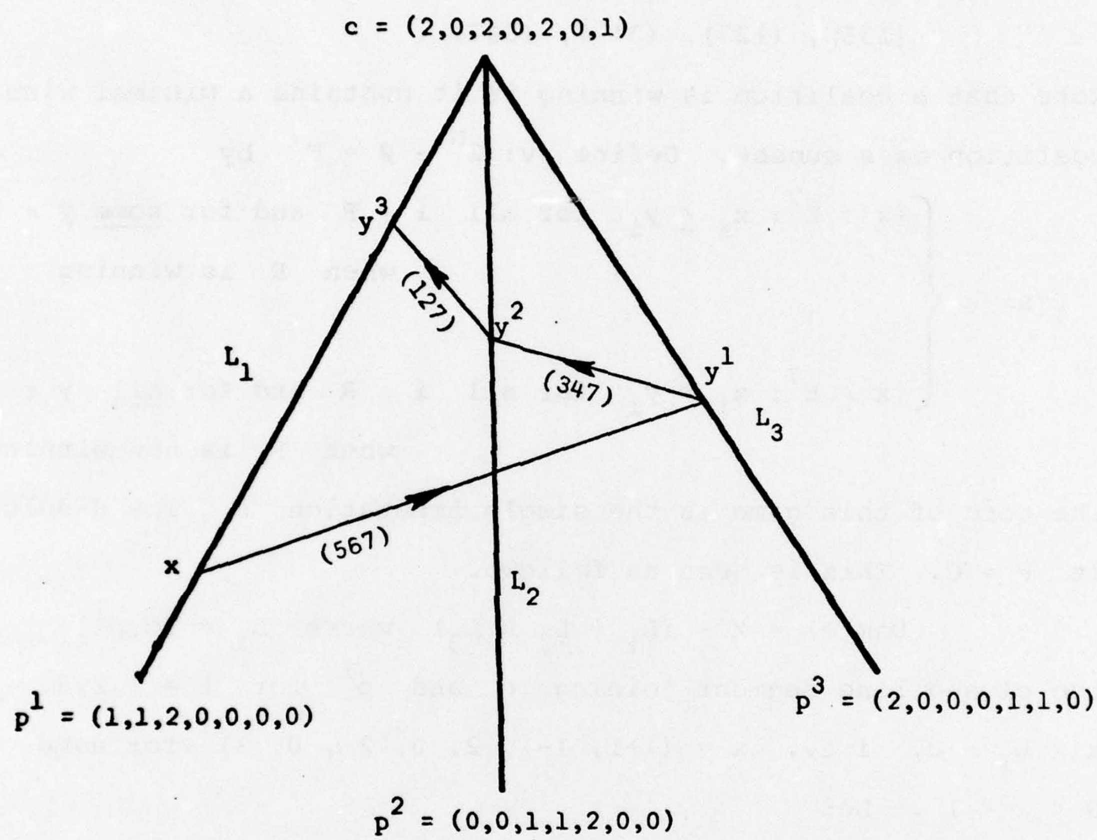


Figure 7. A 7-person non side payment game with no stable sets.

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Several solution concepts have been defined for abstract games. Some of these are the core due to Gillies ~~[3]~~ and Shapley, the von Neumann-Morgenstern stable sets due to von Neumann and Morgenstern ~~[15]~~, and the subsolutions due to Roth ~~[11]~~. These solutions concepts are rather static in nature. In this paper, ~~we~~ proposes a new solution concept for abstract games called the dynamic solution that reflects the dynamic aspects of negotiation among players. Some properties of the dynamic solutions are studied. Also, the dynamic solution of abstract games arising from n-person cooperative games in characteristic function form is investigated.

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